ALGEBRAIC STRUCTURE

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Algebraic Structure

An **algebraic structure** consists of a set *A* , a collection of binary operations on *A* and a finite set of identities, known as axioms, that these operations must satisfy.

1. Examples of algebraic structures with a single underlying set include groups, rings, fields, and lattices.

2. Examples of algebraic structures with two underlying sets include vector spaces, modules, and algebras.

3. In particular we have the set of all integers denoted by Z under binary operation'+" is an algebraic structure,

Similarly the set of all rational numbers, Q under binary operation 'x' is am algebraic structure.

Basic Definition:

1. **Union:** The union of two sets A and B written as AUB, $\{x/x\epsilon\ A\ or\ x\ \epsilon B\}$

2. **Intersection:** The intersection of two sets A and B written as $A \cap B, \{x/x\epsilon A$ and $x\ \epsilon B\}$

3. **Disjoint set:**

Two sets are said to be disjoint if their intersection is empty i.e., the null set.

Example: If A is a positive integer and B is a negative integer then the intersection of A and B is empty set

Equivalence Relations:

The binary relation~ on A is said to be an equivalence relation on A for all a,b,c in A

i) $a \sim a$ is called reflexive

ii) $a \sim b$ and $b \sim a$ is called symmetry

iii) $a \sim b$ and $b \sim c$ then $a \sim c$ is called transitivity.

GROUP:

A group is a set, G, together with an operation that combines any two elements a and b to form another element, denoted by $a \cdot b$ or *ab and satisfies the following axioms.*

1. Closure

For all a, b in G, the result of the operation, $a \cdot b$, is also in G.

2. Associativity

For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

3. Identity element

There exists an element e in G such that, for every element a in G, the equation $e \cdot a = a \cdot e = a$ holds. Such an element is unique and thus one speaks of *the* identity element.

4. Inverse element

For each a in G, there exists an element b in G, commonly denoted a^{-1} (or $-a$, if the operation is denoted "+"), such that $a \cdot b = b \cdot a = e$, where e is the identity element.

Example:

The set of integers is a group

$\{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$ under binary operation addition.

• **Closure:** For any two integers a and b, the sum $a + b$ is also an integer. That is, addition of integers always yields an integer. It satisfies closure property.

• **Associative :** For all integers a, b and c, $(a + b) + c = a + (b + c)$. It satisfies a property known as *associativity*.

• **Identity :** If a is any integer, then $0 + a = a + 0 = a$. Zero is called the *identity element* of addition because adding it to any integer gives the same integer.

•**Inverse :** For every integer a, there is an integer -a such that a -a = $-a + a = 0$. The integer $-a$ is called the *inverse element* of the integer a

Therefore The integers, together with the operation +, forms a group.

Finite Group: A group is said to be finite if contains finite number of elements.

Infinite Group: A group is said to be infinite if contains infinite number of elements.

Example:

1. The set S = {1,-1} is a finite group under binary operation multiplication

2.The set of all integers is a infinite group under the binary operation addition

Order of the group:

The number of elements in a group is called as the order of the group.

Example: The set A = {1,-1} is a group under multiplication and the order is 2.

Trivial group : A group that contains only one element is called as Trivial group.

Example: The set B = {0} is a trivial group under binary operation addition.

Abelian Group:

A group (G,*) is said to be abelian group if a*b = b* a for all a , b ϵ G.

Binary operation : Let S be a non empty set then an operation ‘.’ is said to be binary operation on S. If ‘.’ is function from binary operation if a . b ϵ S for all a , b ϵ S.

Example: Using addition and multiplication or binary operation

i) Z, the set of all integers

Ii) N. the set of all natural numbers {1,2,3,4,5…….}

ii) Q , set of all rational numbers {-∞ to ∞} p/q if q not equal to 0.

Semi group

A **semigroup** is an algebraic structure consisting of a set together with an associative binary operation.

The binary operation of a semigroup is most often denoted multiplicatively: $x \cdot y$ or simply $x\,y$

Associativity is formally expressed as

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y and z in the semigroup.

Sub group: A subset H of a group G is called a subgroups of G if

i) whenever a ϵ H , b ϵ H . The product ab ϵ H

ii) e ϵ H

iii) For each a ϵ H its inverse a^{-1}in G also belongs to H.

A non empty subset H of a group G is called a subgroup of G if H itself form a group with respect to the binary operation defined in G.

Example: The group of integer under addition forms a subgroup of G.

Group

A group is a set, G, together with an operation that combines any two elements a and b to form another element, denoted by $a \cdot b$ or *ab and satisfies the following axioms.*

1. Closure

For all a, b in G, the result of the operation, $a \cdot b$, is also in G.

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For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

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Cancellation Law

Let G be a group and a, b, c elements of G .Then ab=ac implies b=c (left cancellation law), Similarly ba = ca implies b=c (Right cancellation law)

Proof: **Consider the relation**

ab=ac

$a^{-1}(ab) = a^{-1}(ac)$

By associative

$(a^{-1}a)b = (a^{-1}a)c$

eb=ec

b=c (Left cancellation law)

ii)

ba =ca

Multiply by a^{-1} on both sides

$(ba)a^{-1} = (ca)a^{-1}$

$b(aa^{-1}) =c(aa^{-1})$

be=ce

b=c(right cancellation law)

Theorem :2

Let G be a group and a,b elements of G then the equations ax=b an dya=b have unique solution in G.

Proof:

Consider the equation

ax=b

Multiply by a^{-1} on both sides

$a^{-1(ax) =} a^{-1}b$

$(a^{-1}a)x = a^{-1}b$

$ex = a^{-1}b$

$x = a^{-1}b$

now ya = b

Multiply by a^{-1} on both sides

$(ya)\, a^{-1} = b\, a^{-1}$

$y(a\, a^{-1}) = b\, a^{-1}$

$y = b\, a^{-1}$

Therefore x and y exists

To prove uniqueness:

Let x_1 and x_2 be two solutions of ax = b for x

Then ax_1 =b for x_1 and $ax_2 = ax_2$

$X_1=x_2$ (left cancellation law)

Let y_1 and y_2 be two cancellation of ya = b for y then y_1a= b andy_2 =b

$y_1a = y_2a$

$y_1 = y_2$ (Right cancellation law)

Hence the equations ax=b and ya = b have unique solution of G.

Cyclic group

- If a group G contains an element ‘a’ such that every element of G is of the form ‘$a^{k\prime}$ for some integer k we say that G is a cyclic group and that a is called as generator.
- If G is a group under addition then G is called Cyclic if there exists an element ‘a’ in G such that every element of G is of the form ka for some integer k.
- There may be more than one generator in a cyclic group.
- If G is a cyclic group generator by a then we write G = (a)
- **Example:** Consider the group G ={1,-1,i,-i} under multiplication generator are i and –i.

Co sets

Let H be a subgroup of a group G let ‘a’ be an arbitrary element of G then there a H = {ah/h ϵH} is called a left co sets of H in G.

Any set of the type Ha = {ha/h ϵ H} is called a right co set of H in G.

Index of H in G

If H is a subgroup of a group G, then the number of disjoint left co set (right) of H in G is called the index of H in G and is denoted by [G:H]