NAZARETH COLLEGE OF ARTS AND SCIENCE

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DEPARTMENT OF MATHEMATICS

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Subject Name: Integral Calculus

Topic Name: Beta and Gamma Functions

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Unit-3: Beta and Gamma function

Gamma function

The integral $\int_0^{\infty} e^{-x} x^{n-1} dx$ $(n>0)$ its called a gamma function and in symbol it is denoted by

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx.$$

Here, n is any positive real number.

Gamma function is also called Eulerian integral of first kind.

Beta function:

The integral $\int_0^1 x^{m-1} (1-x)^{n-1} x\, dx$ for $m>0, n>0$ is called beta function and is denoted by $\beta(m,n)$.

(i.e.,) $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$

Property 1:

$$\Gamma(n-1) = n\Gamma n \text{ (or) } n!$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx.$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1}\, dx.$$

$$= \int_0^\infty e^{-x} x^{n}\, dx.$$

$u = x^n$ $\quad$ $dv = \int e^{-x}\, dx.$

$du = n x^{n-1}\, dx$ $\quad$ $v = -e^{-x}$

$[\because \int u\, dv = uv - \int v\, du]$

$$\Gamma(n+1) = \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty -e^{-x} n x^{n-1}\, dx$$

$$= 0 + n\int_0^\infty e^{-x} x^{n-1}\, dx \quad [\because e^{-\infty} = 0,\ e^{\infty} = \infty]$$

$$\Gamma(n+1) = n\Gamma n$$

To find $n!$

$$\therefore \Gamma(n+1) = n\Gamma(n)$$

Replacing n by $(n-1)$

$$\Gamma(n-1+1) = (n-1)\Gamma(n-1)$$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$= (n-1)\Gamma(n-1-1+1)$$

$$= (n-1)\Gamma((n-2)+1)$$

$$= (n-1)(n-2)\Gamma(n-2)$$

$$= (n-1)(n-2)\Gamma(n-2+1-1)$$

$$= (n-1)(n-2)\Gamma((n-3)+1)$$

$$= (n-1)(n-2)(n-3)\Gamma(n-3)$$

$$\vdots$$

$$= \Gamma(1)$$

$= \Gamma 1$

Multipling all these

$\Gamma n+1 = n(n-1) \cdots 2\Gamma 1$

$= n! \, \Gamma 1$

from the definition $\left[\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx\right]$

$\Gamma 1 = \int_0^\infty e^{-x} x^{1-1} dx$

$= \int_0^\infty e^{-x} dx \quad [\because x^0 = 1]$

$= \left. \frac{e^{-x}}{-1} \right]_0^\infty = -\left[e^{-\infty} - e^0\right] \quad [\because e^{-\infty} = 0]$

$= -[0-1]$

$= 1.$

$\therefore \Gamma n+1 = n!$

Property: 2.

Another form of gamma function

$$\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dx.$$

Proof.

By definition

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx.$$

Put $x = y^2$, $dx = 2y\,dy$.

when $x = 0$, $y = 0$.

when $x = \infty$, $y = \infty$.

$$\Gamma n = \int_0^\infty e^{-y^2} (y^2)^{n-1}\, 2y\,dy.$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-2} y^1\, dy$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-2+1}\, dy.$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy.$$

Property: 3

$$\beta(m,n) = \beta(n,m).$$

Proof

from the definition of Beta function.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

$$\left[\text{using the property } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{m-1}\left(1-(1-x)\right)^{n-1} dx.$$

$$= \int_0^1 (1-x)^{m-1}(1-1+x)^{n-1}\,dx.$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx.$$

$$\beta(m,n) = \beta(n,m)$$

Property : 4

The Second form of Beta function

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta .$$

Proof

By definition,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx .$$

Put $x = \sin^2\theta$ —(1)

$dx = 2\sin\theta\cos\theta \, d\theta$.

When $x = 0$ in Eq (1) $0 = \sin^2\theta$.

$\Rightarrow \sin\theta = 0$

$\theta = \sin^{-1}(0)$

$\theta = 0$

When $x = 1$ in Eq (1) $1 = \sin^2\theta$

$\Rightarrow 1 = \sin\theta$

$\theta = \sin^{-1}(1)$

$\Rightarrow \theta = \pi/2$.

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \, (\cos^2\theta)^{n-1} \sin\theta\cos\theta \, d\theta .$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \cos^{2n-2+1}\theta \, d\theta .$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta .$$

Property : 5

The third form of Beta function

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy.$$

Proof

By definition

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

$$\text{Put } x = \frac{y}{1+y} \quad \text{——— (1)}$$

$$x(1+y) = y$$

$$x + xy = y.$$

$$x = y - xy$$

$$x = (1-x)y$$

$$\therefore y = \frac{x}{1-x}. \quad \text{——— (2).}$$

$$x = \frac{y}{1+y}. \qquad \left[\frac{u}{v} = \frac{du\,v - u\,dv}{v^2}\right]$$

$$dx = \frac{(1+y)\cdot 1 - y(1)}{(1+y)^2}\,dy$$

$$= \frac{1+\cancel{y}-\cancel{y}}{(1+y)^2}\,dy$$

$$dx = \frac{1}{(1+y)^2}\,dy$$

when $x=0$ in (2) $y = 0$.

when $x=1$ in (2) $y = \infty$

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \cdot \left(1 - \frac{y}{1+y}\right)^{n-1} \cdot \frac{dy}{(1+y)^2}.$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \left(\frac{1+y-y}{1+y}\right)^{n-1} \cdot \frac{1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2} \cdot dy.$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1+n-1+2}} \, dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} \, dy$$

Property: 6

$$\int_0^\infty e^{-ay} y^{n-1} dy = \frac{\Gamma n}{a^n}.$$

Proof

Put $x = ay$ ——(1)

$dx = a\,dy$.

when $x = 0$ in (1) $y = 0$.

when $x = \infty$ in (1) $y = \infty$

w.k.t

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx.$$

$$= \int_0^\infty e^{-ay} (ay)^{n-1} a\,dy$$

$$= \int_0^\infty e^{-ay} a^{n-1} \cdot y^{n-1} \cdot a\,dy$$

$$= \int_0^\infty e^{-ay} y^{n-1} a^{n-1+1} dy$$

$$\Gamma n = a^n \int_0^\infty e^{-ay} y^{n-1} dy$$

$$\therefore \frac{\Gamma n}{a^n} = \int_0^\infty e^{-ay} y^{n-1} dy.$$

Property : 7

Relation between Beta and Gamma function.

Statement : $B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Proof:

By property (2).

$$\Gamma m = 2\int_0^\infty e^{-y^2} y^{2m-1} dy$$

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx.$$

$$\Gamma m \, \Gamma n = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} y^{2m-1} x^{2n-1} dx\,dy.$$

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

$x^2 = r^2\cos\theta$, $y^2 = r^2\sin\theta$.

$x^2+y^2 = r^2(\sin^2\theta + \cos^2\theta)$

$r = \sqrt{x^2+y^2}.$

$$r = \sqrt{x^2+y^2}.$$

when $x=0, y=0, r=0$.

$x=\infty, y=\infty, r=\infty$.

$\therefore r = 0$ to ∞.

$$y/x = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \frac{y}{x} = \tan\theta.$$

$$\Rightarrow \theta = \tan^{-1}(y/x).$$

when $x=0, y=0, \quad \theta = \tan^{-1}(0) \Rightarrow \theta = 0.$

$x=\infty, y=\infty, \quad \theta = \tan^{-1}(\infty) \Rightarrow \theta = \pi/2.$

$$\Gamma m \Gamma n = 4\int_0^{\pi/2}\int_0^{\infty} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)} \cdot (r\sin\theta)^{2m-1} \cdot (r\cos\theta)^{2n-1} r\,dr\,d\theta.$$

$$= 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} \cdot r^{2m-1} \cdot \sin^{2m-1}\theta \; r^{2n-1} \cdot \cos^{2n-1}\theta \; r\,dr\,d\theta.$$

$$= \left[2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta\right] \cdot \left[2\int_0^{\infty} e^{-r^2} r^{2m-1+2n-1+1} \, dr\right]$$

$$= \beta(m,n)\, 2\int_0^{\infty} e^{-r^2} r^{2m+2n-1} \, dr \quad [\text{By Property } 4]$$

$$= \beta(m,n)\left[2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr\right]$$

$$\Gamma m\, \Gamma n = \beta(m,n)\, \Gamma(m+n) \quad [\text{By property } 2]$$

$$\therefore \beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}.$$

Property: 8

$$\beta(m,n) = \beta(m,n+1) + \beta(m+1,n)$$

Proof:

$$R.H.S = \beta(m,n+1) + \beta(m+1,n)$$

$$= \frac{\Gamma m \, \Gamma n+1}{\Gamma m+n+1} + \frac{\Gamma m+1 \, \Gamma n}{\Gamma m+1+n} \quad \left[\because \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \right]$$

$$= \frac{\Gamma m \, n \Gamma n}{\Gamma m+n+1} + \frac{m \Gamma m \, \Gamma n}{\Gamma m+n+1}$$

$$= \frac{\Gamma m \, n \Gamma n + m \Gamma m \, \Gamma n}{\Gamma m+n+1}$$

$$= \frac{\Gamma m \, \Gamma n \, (n+m)}{(m+n) \Gamma m+n}$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$= \beta(m,n)$$

$$= L.H.S$$

Hence proved.