NAZARETH COLLEGE OF ARTS AND SCIENCE

Affiliated to University of Madras

Re - accredited by NAAC with 'B' grade

DEPARTMENT OF MATHEMATICS

Name of the Faculty : B.Uma

Designation : Assistant Professor

Subject : Trigonometry

Topic: Expansion of Sin$\boldsymbol{\theta}$ & $\boldsymbol{Cos\ \theta}$

Expansion of $\sin^n\theta$ and $\cos^n\theta$

$$\text{Let } x = \cos\theta + i\sin\theta \qquad \dots (1)$$

$$\therefore \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta}$$

$$= (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos\theta - i\sin\theta \qquad \dots (2) \quad \text{[By DeMovire's theorem]}$$

$$\text{Also} \quad x^n = (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta \qquad \dots (3)$$

$$\frac{1}{x^n} = \frac{1}{\cos n\theta + i\sin n\theta}$$

$$= (\cos n\theta + i\sin n\theta)^{-1}$$

$$= \cos n\theta - i\sin n\theta \quad \dots (4)$$

(1) + (2)

$$x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta \quad \dots (5)$$

(1) − (2)

$$x - \frac{1}{x} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$$

$$= \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$

$$= 2i\sin\theta \quad \dots (6)$$

(3) + (4)

$$x^n + \frac{1}{x^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta \quad \dots (7)$$

(3) − (4)

$$x^n - \frac{1}{x^n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$$

$$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$

$$= 2i\sin n\theta \quad \dots (8)$$

1. Expand $\cos^5\theta$

Solution :

$$\text{Let } x = \cos\theta + i\sin\theta \quad \dots (1)$$

$$\frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta}$$

$$= (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos\theta - i\sin\theta \quad \dots (2)$$

$$\text{Also } x^n = (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta \quad \dots (3)$$

$$\frac{1}{x^n} = \frac{1}{\cos n\theta + i\sin n\theta}$$

$$= (\cos n\theta + i\sin n\theta)^{-1}$$

$$= \cos n\theta - i\sin n\theta \quad \dots (4)$$

(1) + (2)

$$x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta \quad \dots (5)$$

(1) − (2)

$$x - \frac{1}{x} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$$

$$= \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$

$$= 2i\sin\theta \quad \dots (6)$$

(3) + (4)

$$x^n + \frac{1}{x^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta \qquad \text{... (7)}$$

(3) − (4)

$$x^n - \frac{1}{x^n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$$

$$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$

$$= 2i\sin n\theta \qquad \text{... (8)}$$

To find $\cos^5\theta$, we have to expand $\left(x + \frac{1}{x}\right)^5$

$$\left(x + \frac{1}{x}\right)^5 = (2\cos\theta)^5 = 2^5\cos^5\theta$$

$$\therefore\ 2^5\cos^5\theta = \left(x + \frac{1}{x}\right)^5$$

$$= x^5 + 5C_1 x^{5-1}\cdot\frac{1}{x} + 5C_2\cdot x^{5-2}\cdot\left(\frac{1}{x}\right)^2 + 5C_3 x^{5-3}\cdot\left(\frac{1}{x}\right)^3$$

$$+\ 5C_4 x^{5-4}\cdot\left(\frac{1}{x}\right)^4 + 5C_5 x^{5-5}\cdot\left(\frac{1}{x}\right)^5$$

$$= x^5 + 5x^4\left(\frac{1}{x}\right) + 10x^3\left(\frac{1}{x^2}\right) + 10x^2\left(\frac{1}{x^3}\right) + 5x\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$

$$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$$

$$= x^5 + \frac{1}{x^5} + 5x^3 + \frac{5}{x^3} + 10x + \frac{10}{x}$$

$$2^5 \cos^5 \theta = \left(x^5 + \frac{1}{x^5}\right) + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$$

Using (7)

$$2^5 \cos^5 \theta = 2 \cos 5\theta + 5\,(2 \cos 3\theta) + 10\,(2 \cos \theta)$$

$$2^5 \cos^5 \theta = 2 \cos 5\theta + 10 \cos 3\theta + 20 \cos \theta$$

$$\cos^5 \theta = \frac{1}{2^5}\,[2 \cos 5\theta + 10 \cos 3\theta + 20 \cos \theta]$$

$$= \frac{2}{2^5}\,[\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta]$$

$$= \frac{1}{2^4}\,[\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta]$$

$$= \frac{1}{16}\,[\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta]$$