

# **Vector Spaces**

## **Linear Independence, Bases and Dimension**

# Premise

- We move to a higher level of abstraction
- A vector space is a set with an addition and scalar multiplication that behave appropriately, that is, like  $\mathbb{R}^n$
- Imagine a vector space as a class of a generic type (template) in object oriented programming, equipped with two operations.

# Vector Spaces

## Definition (Vector Space)

A (real) **vector space**  $V$  is a non-empty set equipped with an **addition** and a **scalar multiplication** operation such that for all  $\alpha, \beta \in \mathbb{R}$  and all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ :

1.  $\mathbf{u} + \mathbf{v} \in V$  (closure under addition)
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  (commutative law for addition)
3.  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$  (associative law for addition)
4. there is a single member  $\mathbf{0}$  of  $V$ , called the **zero vector**, such that for all  $\mathbf{v} \in V$ ,  $\mathbf{v} + \mathbf{0} = \mathbf{v}$
5. for every  $\mathbf{v} \in V$  there is an element  $\mathbf{w} \in V$ , written  $-\mathbf{v}$ , called the **negative** of  $\mathbf{v}$ , such that  $\mathbf{v} + \mathbf{w} = \mathbf{0}$
6.  $\alpha \mathbf{v} \in V$  (closure under scalar multiplication)
7.  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$  (distributive law)
8.  $(\alpha + \beta)\mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v}$  (distributive law)
9.  $\alpha(\beta \mathbf{v}) = (\alpha\beta)\mathbf{v}$  (associative law for vector multiplication)
10.  $1\mathbf{v} = \mathbf{v}$

# Examples

- set  $\mathbb{R}^n$
- but the set of objects for which the vector space defined is valid are more than the vectors in  $\mathbb{R}^n$ .
- set of all functions  $F : \mathbb{R} \rightarrow \mathbb{R}$ .  
We can define an addition  $f + g$ :

$$(f + g)(x) = f(x) + g(x)$$

and a scalar multiplication  $af$ :

$$(af)(x) = af(x)$$

- Example:  $x + x^2$  and  $2x$ . They can represent the result of the two operations.
- What is  $-f$  ? and the zero vector?

The axioms given are minimum number needed. Other properties can be derived:  
For example:

$$(-1)x = -x$$

Proof:

$$0 = 0x = (1 + (-1))x = 1x + (-1)x = x + (-1)x$$

Adding  $-x$  on both sides:

$$-x = -x + 0 = -x + x + (-1)x = (-1)x$$

which proves that  $-x = (-1)x$ .

Try the same with  $-f$ .

# Examples

- $V = \{0\}$
- the set of all  $m \times n$  matrices
- the set of all infinite sequences of real numbers,  $\mathbf{y} = \{y_1, y_2, \dots, y_n, \dots\}, y_i \in \mathbb{R}$ .  
 ( $\mathbf{y} = \{y_n\}, n \geq 1$ )
  - addition of  $\mathbf{y} = \{y_1, y_2, \dots, y_n, \dots\}$  and  $\mathbf{z} = \{z_1, z_2, \dots, z_n, \dots\}$  then:  

$$\mathbf{y} + \mathbf{z} = \{y_1 + z_1, y_2 + z_2, \dots, y_n + z_n, \dots\}$$
  - multiplication by a scalar  $a \in \mathbb{R}$ :  

$$a\mathbf{y} = \{ay_1, ay_2, \dots, ay_n, \dots\}$$
- set of all vectors in  $\mathbb{R}^3$  with the third entry equal to 0 (verify closure):

$$W = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

# Linear Combinations

## Definition (Linear Combination)

For vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  in a vector space  $V$ , the vector

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k$$

is called a **linear combination** of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ . The scalars  $a_i$  are called **coefficients**.

- To find the coefficients that given a set of vertices express by linear combination a given vector, we solve a system of linear equations.
- If  $F$  is the vector space of functions from  $\mathbb{R}$  to  $\mathbb{R}$  then the function  $f : x \mapsto 2x^2 + 3x + 4$  can be expressed as a linear combination of:  
 $g : x \mapsto x^2, h : x \mapsto x, k : x \mapsto 1$  that is:  
$$f = 2g + 3h + 4k$$
- Given two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , is it possible to represent any point in the Cartesian plane?

# Subspaces

## Definition (Subspace)

A subspace  $W$  of a vector space  $V$  is a non-empty subset of  $V$  that is itself a vector space under the same operations of addition and scalar multiplication as  $V$ .

## Theorem

Let  $V$  be a vector space. Then a non-empty subset  $W$  of  $V$  is a subspace if and only if both the following hold:

- for all  $u, v \in W$ ,  $u + v \in W$   
( $W$  is closed under addition)
- for all  $v \in W$  and  $a \in \mathbb{R}$ ,  $av \in W$   
( $W$  is closed under scalar multiplication)

ie, all other axioms can be derived to hold true

## Example

- The set of all vectors in  $\mathbb{R}^3$  with the third entry equal to 0.
- The set  $\{0\}$  is not empty, it is a subspace since  $0 + 0 = 0$  and  $a0 = 0$  for any  $a \in \mathbb{R}$ .

## Example

In  $\mathbb{R}^2$ , the lines  $y = 2x$  and  $y = 2x + 1$  can be defined as the sets of vectors:

$$S = \begin{pmatrix} x \\ y \end{pmatrix} \mid y = 2x, x \in \mathbb{R} \quad U = \begin{pmatrix} x \\ y \end{pmatrix} \mid y = 2x + 1, x \in \mathbb{R}$$

$$S = \{x \mid x = tv, t \in \mathbb{R}\}$$

$$U = \{x \mid x = p + tv, t \in \mathbb{R}\}$$

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

### Example (cntd)

1. The set  $S$  is non-empty, since  $\mathbf{0} = 0\mathbf{v} \in S$ .
2. closure under addition:

$$\mathbf{u} = s \sum_{i=1}^2 \mathbf{v}_i \in S, \quad \mathbf{w} = t \sum_{i=1}^2 \mathbf{v}_i \in S, \quad \text{for some } s, t \in \mathbb{R}$$

$$\mathbf{u} + \mathbf{w} = s\mathbf{v} + t\mathbf{v} = (s+t)\mathbf{v} \in S \text{ since } s+t \in \mathbb{R}$$

3. closure under scalar multiplication:

$$\mathbf{u} = s \sum_{i=1}^2 \mathbf{v}_i \in S \quad \text{for some } s \in \mathbb{R}, \quad a \in \mathbb{R}$$

$$a\mathbf{u} = a(s\mathbf{v}) = (as)\mathbf{v} \in S \text{ since } as \in \mathbb{R}$$

Note that:

- $\mathbf{u}, \mathbf{w}$  and  $a \in \mathbb{R}$  must be arbitrary

Note that:

- proving just one of the above counterexamples is enough to show that  $U$  is not a subspace
- it is sufficient to make them fail for particular choices
- a good place to start is checking whether  $0 \in S$ . If not then  $S$  is not a subspace

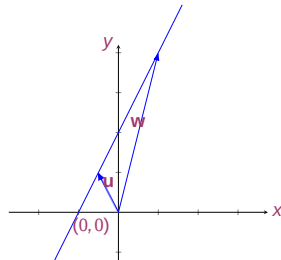
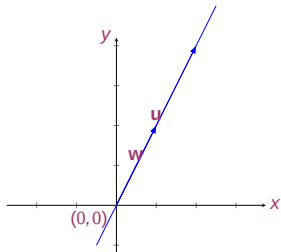
## Theorem

A non-empty subset  $W$  of a vector space is a subspace if and only if for all  $\mathbf{u}, \mathbf{v} \in W$  and all

$\alpha, \beta \in \mathbb{R}$ , we have  $\alpha\mathbf{u} + \beta\mathbf{v} \in W$ .

That is,  $W$  is closed under linear combination.

Geometric  
interpretation:



~ The line  $y = 2x + 1$  is an affine subset, a „translation“ of a subspace

# Null space of a Matrix is a Subspace

## Theorem

For any  $m \times n$  matrix  $A$ ,  $N(A)$ , ie, the solutions of  $Ax = 0$ , is a **subspace** of  $\mathbb{R}^n$

## Proof

1.  $A0 = 0 \Rightarrow 0 \in N(A)$
2. Suppose  $u, v \in N(A)$ , then  $u + v \in N(A)$ :  
$$A(u + v) = Au + Av = 0 + 0 = 0$$
3. Suppose  $u \in N(A)$  and  $a \in \mathbb{R}$ , then  $au \in N(A)$ :  
$$A(au) = A(au) = aAu = a0 = 0$$



The set of solutions  $S$  to a general system  $Ax = b$  is not a

**subspace** of  $\mathbb{R}^n$  because  $0 \notin S$

# Affine subsets

## Definition (Affine subset)

If  $W$  is a **subspace** of a **vector space**  $V$  and  $\mathbf{x} \in V$ , then the set  $\mathbf{x} + W$  defined by

$$\mathbf{x} + W = \{\mathbf{x} + \mathbf{w} \mid \mathbf{w} \in W\}$$

is said to be an **affine subset** of  $V$ .

The set of solutions  $S$  to a general system  $A\mathbf{x} = \mathbf{b}$  is an affine **subspace**, indeed recall that if  $\mathbf{x}_0$  is any solution of the system

$$S = \{\mathbf{x}_0 + \mathbf{z} \mid \mathbf{z} \in N(A)\}$$

# Range of a Matrix is a Subspace

## Theorem

For any  $m \times n$  matrix  $A$ ,  $R(A) = \{Ax \mid x \in \mathbb{R}^n\}$  is a **subspace** of  $\mathbb{R}^m$

## Proof

1.  $A\mathbf{0} = \mathbf{0} \implies \mathbf{0} \in R(A)$
2. Suppose  $\mathbf{u}, \mathbf{v} \in R(A)$ , then  $\mathbf{u} + \mathbf{v} \in R(A)$ :  
...
3. Suppose  $\mathbf{u} \in R(A)$  and  $a \in \mathbb{R}$ , then  $a\mathbf{u} \in R(A)$ :  
...

# Linear Span

- If  $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k$  and  $\mathbf{w} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \dots + \beta_k \mathbf{v}_k$ , then  $\mathbf{v} + \mathbf{w}$  and  $s\mathbf{v}$ ,  $s \in \mathbb{R}$  are also linear combinations of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .
- The set of all linear combinations of a given set of vectors of a **vector space**  $V$  forms a **subspace**:

## Definition (Linear span)

Let  $V$  be a **vector space** and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ . The **linear span** of  $X = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is the set of all linear combinations of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ , denoted by  $\text{Lin}(X)$ , that is:

$$\text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}) = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k \mid \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}\}$$

## Theorem

If  $X = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is a set of vectors of a **vector space**  $V$ , then  $\text{Lin}(X)$  is a **subspace** of  $V$  and is also called the **subspace spanned by**  $X$ .

It is the smallest **subspace** containing the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

## Example

- $\text{Lin}(\{\mathbf{v}\}) = \{a\mathbf{v} \mid a \in \mathbb{R}\}$  defines a line in  $\mathbb{R}^n$ .
- Recall that a plane in  $\mathbb{R}^3$  has two equivalent representations:

$$ax + by + cz = d \quad \text{and} \quad \mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}$$

where  $\mathbf{v}$  and  $\mathbf{w}$  are non  
parallel.

–If  $d = 0$  and  $\mathbf{p} = \mathbf{0}$ , then  
 $\{\mathbf{x} \mid \mathbf{x} = s\mathbf{v} + t\mathbf{w}, s, t, \in \mathbb{R}\} = \text{Lin}(\{\mathbf{v}, \mathbf{w}\})$

and hence a **subspace** of  $\mathbb{R}^n$ .

–If  $d \neq 0$ , then the plane is not a **subspace**. It is an **affine subset**, a translation of a **subspace**.

(recall that one can also show directly that a subset is a **subspace** or not)

# Spanning Sets of a Matrix

## Definition (Column space)

If  $A$  is an  $m \times n$  matrix, and if  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$  denote the columns of  $A$ , then the **column space** or **range** of  $A$  is

$$CS(A) = R(A) = \text{Lin}(\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k\})$$

and is a **subspace** of  $\mathbb{R}^m$

## . Definition (Row space)

If  $A$  is an  $m \times n$  matrix, and if  $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_k$  denote the rows of  $A$ , then the **row space** of  $A$  is

$$RS(A) = \text{Lin}(\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_k\})$$

and is a **subspace** of  $\mathbb{R}^n$ .

- If  $A$  is an  $m \times n$  matrix, then for any  $\mathbf{r} \in RS(A)$  and any  $\mathbf{x} \in N(A)$ ,  $(\mathbf{r}, \mathbf{x}) = 0$ ; that is,  $\mathbf{r}$  and  $\mathbf{x}$  are **orthogonal**,  $RS(A) \perp N(A)$ . (hint: look at  $A\mathbf{x} = \mathbf{0}$ )

# Summary

We have seen:

- Definition of **vector space** and **subspace**
- Linear combinations as the main way to work with vector spaces
- Proofs that a given set is a vector space
- Proofs that a given subset of a vector space is a **subspace** or not
- Definition of linear span of set of vectors
- Definition of row and column spaces of a matrix  
 $CS(A) = R(A)$  and  $RS(A) \perp N(A)$

# Linear Independence

## Definition (Linear Independence)

Let  $V$  be a **vector space** and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ . Then  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are **linearly independent** (or form a **linearly independent set**) if and only if the vector equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{0}$$

has the unique solution

$$a_1 = a_2 = \dots = a_k = 0$$

## Definition (Linear Dependence)

Let  $V$  be a **vector space** and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ . Then  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are **linearly dependent** (or form a **linearly dependent set**) if and only if there are real numbers  $a_1, a_2, \dots, a_k$ , not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{0}$$

### Example

In  $\mathbb{R}^2$ , the vectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are linearly independent.

Indeed

$$a \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a + \beta = 0 \\ 2a - \beta = 0 \end{cases}$$

The homogeneous linear system has only the trivial solution,  $a = 0, \beta = 0$ , so linear independence.

### Example

In  $\mathbb{R}^3$ , the following vectors are linearly dependent:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 4 \\ 5 \\ 11 \end{pmatrix}$$

Indeed:  $2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$

## Theorem

The set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq V$  is linearly dependent if and only if at least one vector  $\mathbf{v}_i$  is a linear combination of the other vectors.

### Proof

$\Rightarrow$

If  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  are linearly dependent then

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{0}$$

has a solution with some  $a_i \neq 0$ , then:

$$\mathbf{v}_i = -\frac{a_1}{a_i}\mathbf{v}_1 - \frac{a_2}{a_i}\mathbf{v}_2 - \dots - \frac{a_{i-1}}{a_i}\mathbf{v}_{i-1} - \frac{a_{i+1}}{a_i}\mathbf{v}_{i+1} + \dots - \frac{a_k}{a_i}\mathbf{v}_k$$

which is a linear combination of the other vectors

$\Leftarrow$

If  $\mathbf{v}_i$  is a lin combination of the other vectors, eg

$$\mathbf{v}_i = \beta_1\mathbf{v}_1 + \dots + \beta_{i-1}\mathbf{v}_{i-1} + \beta_{i+1}\mathbf{v}_{i+1} + \dots + \beta_k\mathbf{v}_k$$

then

$$\beta_1\mathbf{v}_1 + \dots + \beta_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + \beta_{i+1}\mathbf{v}_{i+1} + \dots + \beta_k\mathbf{v}_k = \mathbf{0}$$



## Corollary

Two vectors are linearly dependent if and only if at least one vector is a scalar multiple of the other.

$$\mathbf{v}_1 = \begin{bmatrix} \square \\ 2 \\ \square \\ 3 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} \square \\ 1 \\ \square \\ 5 \end{bmatrix}$$

are linearly  
independent

## Theorem

*In a **vector space**  $V$ , a non-empty set of vectors that contains the zero vector is linearly dependent.*

Proof:

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subset V$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{0}\}$$

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_k + a\mathbf{0} = \mathbf{0}, \quad a \neq 0$$

# Uniqueness of linear combinations

## Theorem

If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent vectors in  $V$  and if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_k\mathbf{v}_k$$

then

$$a_1 = b_1, \quad a_2 = b_2, \quad \dots \quad a_k = b_k.$$

- 
- If a vector  $\mathbf{x}$  can be expressed as a linear combination of linearly independent vectors, then this can be done in only one way

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$$

# Testing for Linear Independence in $\mathbb{R}^n$

For  $k$  vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k$$

is equivalent to

$$A\mathbf{x}$$

where  $A$  is the  $n \times k$  matrix whose columns are the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  and  $\mathbf{x} = [a_1, a_2, \dots, a_k]^T$ :

## Theorem

The vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  in  $\mathbb{R}^n$  are **linearly dependent** if and only if the linear system  $A\mathbf{x} = \mathbf{0}$ , where  $A$  is the matrix  $A = [\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_k]$ , has a solution other than  $\mathbf{x} = \mathbf{0}$ .

Equivalently, the vectors are **linearly independent** precisely when the only solution to the system is  $\mathbf{x} = \mathbf{0}$ .

If vectors are linearly dependent, then any solution  $\mathbf{x} \neq \mathbf{0}$ ,  $\mathbf{x} = [a_1, a_2, \dots, a_k]^T$  of  $A\mathbf{x} = \mathbf{0}$  gives a non-trivial linear combination  $A\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{0}$

Recall that  $A\mathbf{x} = \mathbf{0}$  has precisely one solution  $\mathbf{x} = \mathbf{0}$  iff the  $n \times k$  matrix is row equiv. to a row echelon matrix with  $k$  leading ones, ie, iff  $\text{rank}(A) = k$

### Theorem

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ . The set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is linearly independent iff the  $n \times k$  matrix  $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_k]$  has rank  $k$ .

### Theorem

The maximum size of a linearly independent set of vectors in  $\mathbb{R}^n$  is  $n$ .

- $\text{rank}(A) \leq \min\{n, k\}$ , hence  $\text{rank}(A) \leq n \Rightarrow$  when lin. indep.  $k \leq n$ .
- we exhibit an example that has exactly  $n$  independent vectors in  $\mathbb{R}^n$  (there are infinite examples):

$$\begin{matrix} \boxed{1} & \boxed{0} & & \boxed{0} \\ \mathbf{e}_1 = \begin{bmatrix} \boxed{0} \\ \boxed{1} \\ \vdots \\ \boxed{0} \end{bmatrix}, & \mathbf{e}_2 = \begin{bmatrix} \boxed{1} \\ \boxed{0} \\ \vdots \\ \boxed{0} \end{bmatrix}, & \dots, & \mathbf{e}_n = \begin{bmatrix} \boxed{0} \\ \vdots \\ \boxed{1} \end{bmatrix} \end{matrix}$$

This is known as the standard basis of  $\mathbb{R}^n$ .

## Example

e

$$L_1 = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ -1 & 9 & 3 & 9 \\ 0 & 2 & 1 & 1 \end{bmatrix},$$

lin. dep. since  $5 > n = 4$

$$L_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 9 \\ 0 & 2 \end{bmatrix},$$

lin. indep.

$$L_3 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ -1 & 9 & 3 \\ 0 & 2 & 1 \end{bmatrix},$$

lin. dep. since  $\text{rank}(A) = 2$

$$L_4 = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 9 & 3 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix},$$

lin. dep. since  $L_3 \subseteq L_4$

# Linear Independence and Span in $\mathbb{R}^n$

Let  $S = \{v_1, v_2, \dots, v_k\}$  be a set of vectors in  $\mathbb{R}^n$ .

What are the conditions for  $S$  to span  $\mathbb{R}^n$  and be linearly

independent? Let  $A$  be the  $n \times k$  matrix whose columns are the

vectors from  $S$ .

- $S$  spans  $\mathbb{R}^n$  if for any  $v \in \mathbb{R}^n$  the linear system  $Ax = v$  is consistent for all  $v \in \mathbb{R}^n$ . This happens when  $\text{rank}(A) = n$ , hence  $k \geq n$
- $S$  is linearly independent iff the linear system  $Ax = 0$  has a unique solution. This happens when  $\text{rank}(A) = k$ , Hence  $k \leq n$

Hence, to span  $\mathbb{R}^n$  and to be linearly independent, the set  $S$  must have exactly  $n$  vectors and the square matrix  $A$  must have  $\det(A) \neq 0$

## Example

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, v_3 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 5 \\ 1 & 2 & 2 \\ 3 & 5 & 1 \end{vmatrix} = 30 \neq 0$$

# Bases

## Definition (Basis)

Let  $V$  be a **vector space**. Then the subset  $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  of  $V$  is said to be a **basis** for  $V$  if:

1.  $B$  is a linearly independent set of vectors, and
2.  $B$  spans  $V$ ; that is,  $V = \text{Lin}(B)$

## Theorem

*If  $V$  is a **vector space**, then a smallest spanning set is a basis of  $V$ .*

## Theorem

$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis of  $V$  if and only if any  $\mathbf{v} \in V$  is a **unique** linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$

## Example

$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  is the **standard basis** of  $\mathbb{R}^n$ .

the vectors are linearly independent and for any  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ ,

$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$ , ie,

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

## Example

The set below is a basis of

$$\mathbb{R}^2: S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

- any vector  $\mathbf{b} \in \mathbb{R}^2$  is a linear combination of the two vectors in  $S$   
 $\sim \mathbf{Ax} = \mathbf{b}$  is consistent for any  $\mathbf{b}$ .
- $S$  spans  $\mathbb{R}^2$  and is linearly independent

## Example

Find a basis of the **subspace** of  $\mathbb{R}^3$  given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y - 3z = 0 \right\}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x + 3z \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = x \mathbf{v} + z \mathbf{w}, \quad \forall x, z \in \mathbb{R}$$

The set  $\{\mathbf{v}, \mathbf{w}\}$  spans  $W$ . The set is also independent:

$$a\mathbf{v} + \beta\mathbf{w} = \mathbf{0} \Rightarrow a = 0, \beta = 0$$

# Coordinates

## Definition (Coordinates)

If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis of a vector space  $V$ , then any vector  $\mathbf{v} \in V$  can be expressed uniquely as  $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$  then the real numbers  $a_1, a_2, \dots, a_n$  are the coordinates of  $\mathbf{v}$  with respect to the basis  $S$ .

We use the notation

$$[\mathbf{v}]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_S$$

to denote the coordinate vector of  $\mathbf{v}$  in the basis  $S$ .

- We assume the order of the vectors in the basis to be fixed: aka, **ordered basis**
- Note that  $[\mathbf{v}]_S$  is a vector in  $\mathbb{R}^n$ : **Coordinate mapping** creates a one-to-one correspondence between a **general vector space**  $V$  and the familiar vector space  $\mathbb{R}^n$ .

## Example

Consider the two basis of

$\mathbb{R}^2$ :

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[\mathbf{v}]_B = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$[\mathbf{v}]_S = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

In the standard basis the coordinates of  $\mathbf{v}$  are precisely the components of the vector  $\mathbf{v}$ . In the basis  $S$ , they are such that

$$\mathbf{v} = -1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

# Extension of the main theorem

## Theorem

If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent:

1.  $A$  is invertible
2.  $A\mathbf{x} = \mathbf{b}$  has a unique solution for any  $\mathbf{b} \in \mathbb{R}^n$
3.  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution,  $\mathbf{x} = \mathbf{0}$
4. the reduced row echelon form of  $A$  is  $I$ .
5.  $|A| \neq 0$
6. The rank of  $A$  is  $n$
7. The column vectors of  $A$  are a basis of  $\mathbb{R}^n$
8. The rows of  $A$  (written as vectors) are a basis of  $\mathbb{R}^n$

(The last statement derives from  $|A^T| = |A|$ .)

Hence, simply calculating the determinant can inform on all the above facts.

## Exempl

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 4 \\ 5 \\ 11 \end{pmatrix}$$

This set is linearly dependent since  $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$   
 so  $\mathbf{v}_3 \in \text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2\})$  and  $\text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ .  
 The linear span of  $\{\mathbf{v}_1, \mathbf{v}_2\}$  in  $\mathbb{R}^3$  is a plane:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \mathbf{v}_1 + t \mathbf{v}_2 = s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

The vector  $\mathbf{x}$  belongs to the **subspace** iff it can be expressed as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2$ ,  
 that is, if  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}$  are linearly dependent or:

$$|A| = \begin{vmatrix} 1 & 2 & x \\ 2 & 1 & y \\ 3 & 5 & z \end{vmatrix} = 0 \quad \Rightarrow \quad |A| = 7x + y - 3z = 0$$

# Dimension

## Theorem

Let  $V$  be a **vector space** with a basis

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

of  $n$  vectors. Then any set of  $n + 1$  vectors is linearly dependent.

Proof:

Omitted (choose an arbitrary set of  $n + 1$  vectors in  $V$  and show that since any of them is spanned by the basis then the set must be linearly dependent.)

It follows that:

### Theorem

Let a **vector space**  $V$  have a finite basis consisting of  $r$  vectors. Then any basis of  $V$  consists of exactly  $r$  vectors.

### Definition (Dimension)

The number of  $k$  vectors in a finite basis of a **vector space**  $V$  is  $k$  and is the **dimension** of  $V$  denoted by  $\dim(V)$ .

The **vector space**  $V = \{0\}$  is defined to have dimension 0.

- a plane in  $\mathbb{R}^2$  is a two-dimensional **subspace**
- a line in  $\mathbb{R}^n$  is a one-dimensional **subspace**
- a hyperplane in  $\mathbb{R}^n$  is an  $(n - 1)$ -dimensional **subspace** of  $\mathbb{R}^n$
- the **vector space**  $F$  of real functions is an infinite-dimensional **vector space**
- the **vector space** of real-valued sequences is an infinite-dimensional **vector space**.

# Dimension and bases of Subspaces

## Example

The plane  $W$  in  $\mathbb{R}^3$

$$W = \{\mathbf{x} \mid x + y - 3z = 0\}$$

has a basis consisting of the vectors  $\mathbf{v}_1 = [1, 2, 1]^T$  and  $\mathbf{v}_2 = [3, 0, 1]^T$ .

Let  $\mathbf{v}_3$  be any vector  $\mathbf{f} \notin W$ , eg,  $\mathbf{v}_3 = [1, 0, 0]^T$ . Then the set  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis of  $\mathbb{R}^3$ .

# Basis of a Linear Space

If we are given  $k$  vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  in  $\mathbb{R}^n$ , how can we find a basis for  $\text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ ?

We can:

- create an  $n \times k$  matrix (vectors as columns) and find a basis for the column space by putting the matrix in reduced row echelon form

### Definition (Rank and nullity)

The **rank** of a matrix  $A$  is

$$\text{rank}(A) = \dim(R(A))$$

The **nullity** of a matrix  $A$  is

$$\text{nullity}(A) = \dim(N(A))$$

Although **subspaces** of possibly different Euclidean spaces: **Theorem**

If  $A$  is an  $m \times n$  matrix, then

$$\dim(RS(A)) = \dim(CS(A)) = \text{rank}(A)$$

### Theorem (Rank-nullity theorem)

For an  $m \times n$  matrix  $A$

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$(\dim(R(A)) + \dim(N(A)) = n)$$

# Summary

- Linear dependence and independence
- Determine linear dependency of a set of vectors, ie, find non-trivial lin. combination that equal zero
- Basis
- Find a basis for a linear space
- Dimension (finite, infinite)